Relatively Hyperbolic Groups with Rapid Decay Property

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Abstract

We prove that a finitely generated group G that is (strongly) relatively hyperbolic with respect to a collection of finitely generated subgroups $\{H_1, \dots, H_m\}$ has the Rapid Decay property if and only if each H_i, $i = 1, 2, \dots, m$, has the Rapid Decay property.

1 Introduction

Throughout the whole paper, unless stated otherwise, G denotes a finitely generated group, and 1 denotes the neutral element in G.

The L^2-norm of a function x in $l^2(G)$ is denoted by $\|x\|$.

A group G satisfies *the Rapid Decay property* (RD property, in short) if the space of rapidly decreasing functions on G with respect to some length-function is inside the reduced C^*-algebra of G (see Section 2.1 for the precise definition). The RD property is relevant to the Novikov conjecture [CM] and to the Baum-Connes conjecture [L_2].

The main result of the paper is the following.

Theorem 1.1 (Theorem 3.1, Proposition 2.9). *Let G be a group which is (strongly) relatively hyperbolic with respect to some finitely generated subgroups $\{H_1, H_2, \dots, H_m\}$. Then G has the RD property if and only if $\{H_1, \dots, H_m\}$ have the RD property.*

The "only if" part in Theorem 1.1 follows from the more general statement that a subgroup of a group that has the property RD also has RD with respect to the induced length-function [J, Proposition 2.1.1]. The "if" part is more difficult. A proof of it is given in Section 3. In fact we prove that the statement of the theorem still holds if we replace (strong) relative hyperbolicity by a natural weaker version of it, that we call (*)-relative hyperbolicity (see Definition 2.8). At the end of the paper, we shall show how to further weaken property (*) to get an even wider class of groups with RD (see Remark 3.5).

The argument in the proof of Theorem 1.1 is similar in spirit to the arguments used in [RRS], [L_1] and [T]. The idea, in all the mentioned arguments, is to reduce the proof of the inequality (1) to the case when the convolution is performed on some "easier" geodesic triangles. In our case, the "easier" triangles are simply the triangles contained in left cosets gH_i; we can pass from general triangles to triangles in gH_i because of the (*)-relative hyperbolicity.

Some particular cases of Theorem 1.1 have been proven before:

- All hyperbolic groups satisfy the RD property [J, H, C] (this is a particular case of Theorem 1.1 with $m = 1$, $H_1 = \{1\}$).

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- The amalgamated product of two groups A and B with finite amalgamated subgroup F satisfies RD provided A and B satisfy RD [J] (take $G = A *_F B, H_1 = A, H_2 = B$).

- A group G relatively hyperbolic with respect to the subgroups $H_1, \dots, H_m$, has the RD property provided that $H_1, \dots, H_m$ have polynomial growth [ChR]. The fact that polynomial growth implies RD follows from the definition of RD [J].

Not all groups have RD: a group that contains an amenable subgroup of superpolynomial growth (with respect to the word metric of the whole group) does not have RD [J]. Note that it is the only known obstruction to RD.

2 Preliminaries

2.1 Property RD

Recall that a length-function on a group G is a map L from G to the set of non-negative real numbers $\mathbb{R}_+$ satisfying:

(1) $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$;

(2) $L(g) = L(g^{-1})$ for all $g \in G$;

(3) $L(1) = 0$.

Notations: We denote by $\overline{B}_L(r)$ the L-ball of radius r, that is the set $\{g \in G \mid L(g) \leq r\}$ and by $S_L(r)$ the L-sphere of radius r, that is the set $\{g \in G \mid L(g) = r\}$.

We say that a length-function $L_1 \colon G \to \mathbb{R}_+$ *dominates* another length-function $L_2 \colon G \to \mathbb{R}_+$ if there exist $a, b \in \mathbb{R}_+$ such that $L_2 \leq aL_1 + b$. If L_1 dominates L_2 and L_2 dominates L_1 then L_1 and L_2 are said to be *equivalent*.

Remark 2.1. If L_1 dominates L_2 then there exists $c \in \mathbb{R}_+$ such that for every $r \geq 1$ we have $\overline{B}_{L_1}(r) \subset \overline{B}_{L_2}(cr)$.

If G is a finitely generated group, the length-functions corresponding to two finite generating sets are equivalent. All such length-functions are called *word length-functions.*

Lemma 2.2 ([J], Lemma 1.1.4). *If G is a finitely generated group then any word length-function dominates all length-functions on G.*

We first give the traditional analytic version of the RD property with respect to a length-function L, and then present a more geometric one. For every $s \in \mathbb{R}$ *the Sobolev space of order s with respect to L* is the set $H_L^s(G)$ of functions ϕ on G such that the function $(1 + L)^s\phi$ is in $l^2(G)$. The space of *rapidly decreasing functions on G with respect to L* is the set $H_L^\infty(G) = \bigcap_{s\in\mathbb{R}} H_L^s(G)$.

The *group algebra of G*, denoted by $\mathbb{C}G$, is the set of functions with finite support on G, i.e. it is the set of formal linear combinations of elements of G with complex coefficients. We denote by $\mathbb{R}_+G$ its subset consisting of functions taking values in $\mathbb{R}_+$.

With every element $g \in G$ we can associate the linear *convolution operator* $\phi \mapsto g * \phi$ on $l^2(G)$, where

$$g * \phi(h) = \phi(g^{-1}h).$$

This is just the left regular representation of G on $l^2(G)$, it can be extended to a representation of $\mathbb{C}G$ on $l^2(G)$ by linearity. This representation is faithful and every convolution operator induced by an element of $\mathbb{C}G$ is bounded. Therefore we can identify $\mathbb{C}G$ with a subspace in the space of bounded operators $\mathbf{B}(l^2(G))$ on $l^2(G)$. For every $x \in \mathbb{C}G$ we denote by $\|x\|_*$ its operator norm, that is

$$\|x\|_* = \sup\{\|x * \phi\| \,;\, \|\phi\| = 1\}\,.$$

The closure $C_r^*(G)$ of $\mathbb{C}G$ in the operator norm is called *the reduced C^*-algebra of G.*

Definition 2.3. The group G is said to have *the RD property with respect to the length-function* L if the inclusion of $\mathbb{C}G$ into $C_r^*(G)$ extends to a continuous inclusion of $H_L^\infty(G)$ into $C_r^*(G)$.

We recall an equivalent way of defining RD. The following result is a slight modification of Proposition 1.4 in [ChR].

Lemma 2.4. *Let G be a discrete group and let L be a length-function on it. The following statements are equivalent:*

(i) The group G has the RD property with respect to L.

(ii) There exists a polynomial P such that for every $r > 0$, every $x \in \mathbb{R}_+G$ such that x vanishes outside $\overline{B}_L(r)$, and every $\phi \in l^2(G)$ such that $\phi(G) \subseteq \mathbb{R}_+$, we have

$$\|x * \phi\| \leq P(r)\, \|x\| \cdot \|\phi\|\,. \tag{1}$$

Proof. In [ChR, Proposition 1.4] it is proved that (i) is equivalent to (ii) for $x \in \mathbb{R}_+G$ and $\phi \in l^2(G)$. We prove that if (ii) is satisfied for ϕ with $\phi(G) \subseteq \mathbb{R}_+$ then it is satisfied for every $\phi \in l^2(G)$. Let $\phi \in l^2(G)$. Then we can write

$$\phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4),$$

where ϕ_i take values in $\mathbb{R}_+$ and $\|\phi\|^2 = \sum_{i=1}^4 \|\phi_i\|^2$. We have the inequalities

$$\|x * \phi\| \leq \sum_{i=1}^{4} \|x * \phi_i\| \leq P(r)\|x\| \sum_{i=1}^{4} \|\phi_i\| \leq 2P(r)\|x\| \cdot \|\phi\|.$$

□

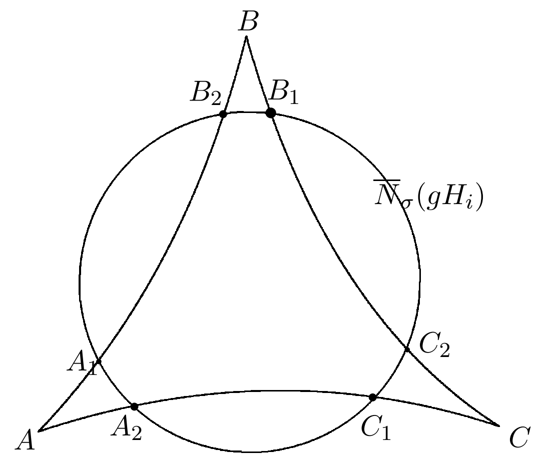
Definition 2.5. A group is said to *have the RD property* if it satisfies the RD property with respect to some length-function.

Remark 2.6. Lemma 2.2 and Remark 2.1 imply that if a finitely generated group satisfies RD with respect to some length-function, then it satisfies RD with respect to a word length-function (for some generating set). Hence, a finitely generated group satisfies RD if and only if it satisfies RD with respect to every word length-function.

In the case of finitely generated groups, it suffices to show (1) for functions with finite support, as shown by the following lemma.

Notation: For a function $f \in l^2(G)$ and a constant $p \geq 0$, f_p denotes the function which coincides with f on $S_L(p)$ and which vanishes outside $S_L(p)$.

Lemma 2.7. *Let G be a finitely generated group and let L be a word length-function on it. The following statements are equivalent:*



(i) The group G has the RD property.

(ii) There exists a polynomial P such that for every $r_1, r_2 \geq 0$, every $p \in [|r_1 - r_2|, r_1 + r_2]$ every $x \in \mathbb{R}_+ G$ with support in $S_L(r_1)$, and every $y \in \mathbb{R}_+ G$ with support in $S_L(r_2)$,

$$\|(x * y)_p\| \leq P(r) \|x\| \cdot \|y\| . \tag{2}$$

Proof. The equivalence is true according to the argument in the proof of Theorem 5, [C, $\S III.5.\alpha$]. $\square$

For details on the RD property we refer to [C], [J], [L_1], [ChR].

2.2 (*)-relative hyperbolicity

Notation: For a subset Y in a metric space we denote by $\overline{\mathcal{N}}_\delta(Y)$ the closed tubular neighborhood of Y, that is $\{x \mid \mathrm{dist}(x, Y) \leq \delta\}$.

Definition 2.8. Let G be a group and let $H_1, ..., H_m$ be subgroups in G. We say that G is (*)-*relatively hyperbolic with respect to* $H_1, ..., H_m$ if there exists a finite generating set S of G, and two constants σ and δ such that the following property holds:

(*) For every geodesic triangle ABC in the Cayley graph of G with respect to S, there exists a coset gH_i such that $\overline{\mathcal{N}}_\sigma(gH_i)$ intersects each of the sides of the triangle, and the entrance (resp. exit) points A_1, B_1, C_1 (resp. B_2, C_2, A_2) of the sides $[A,B], [B,C], [C,A]$ in $\overline{\mathcal{N}}_\sigma(gH_i)$ satisfy
$$\mathrm{dist}(A_1, A_2) < \delta,\ \mathrm{dist}(B_1, B_2) < \delta,\ \mathrm{dist}(C_1, C_2) < \delta .$$

Note that $H_1, ..., H_m$ need not be finitely generated.

The following statement is proved in [DS].

Proposition 2.9 ([DS], Corollary 8.14, Lemma 8.19). *Every group G that is (strongly) relatively hyperbolic with respect to a family of finitely generated subgroups $H_1, ..., H_m$ is (*)-relatively hyperbolic with respect to these subgroups.*

Note that the converse statement is not true. For example, every hyperbolic group G is obviously (*)-relatively hyperbolic with respect to any subgroup $H < G$. But it is well known that for strong relative hyperbolicity to hold H must be quasiconvex.

Problem 2.10. Is it true that every group that is (*)-relatively hyperbolic with respect to certain subgroups $H_1, ..., H_m$ is also strongly relatively hyperbolic with respect to some subgroups $H'_1, ..., H'_n$ such that each H'_i is inside some H_j?

Note that there exists a version of the definition of the strong relative hyperbolicity which also makes sense without $H_1, ..., H_m$ being finitely generated [Os].

Problem 2.11. Can one remove the hypothesis that the subgroups $H_1, ..., H_m$ are finitely generated from the Proposition 2.9 ?

3 Proof of the main result

Theorem 3.1. *Let G be a group which is (*)-relatively hyperbolic with respect to the subgroups $\{H_1, H_2, \dots, H_m\}$. Then G has the RD property if and only if $\{H_1, \dots, H_m\}$ have the RD property with respect to the length-function induced by a word length-function on G.*

Proof. Only the "if" part needs a proof. Consider a finite generating set S of G with respect to which the property (*) is satisfied, and consider the word length-function L on G corresponding to S. By hypothesis and the fact that two word length-functions are equivalent, we may suppose that $H_1, ..., H_m$ have RD with respect to L.

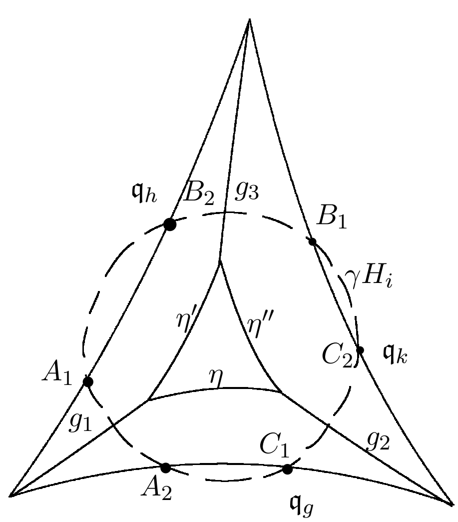
Convention: In what follows we fix two arbitrary positive numbers r_1, r_2 and we fix $p \in [|r_1 - r_2|\,, r_1 + r_2]$.

Let $P_i(r)$ be the polynomial given by Lemma 2.4, (ii), for the group H_i, $i = 1, 2, \dots, m$, and the length function L, and let $P(r) = 1 + \sum_{i=1}^m P_i(r + 2\kappa)^2$, where κ is a constant to be chosen later. We shall prove that for every x with support in $S_L(r_1)$ and every y with support in $S_L(r_2)$, we have the inequality

$$\|(x * y)_p\|^2 \leq Q(r_1)P(r_1)\|x\|^2 \cdot \|y\|^2\,, \tag{3}$$

where Q is a polynomial of degree 3. That will imply inequality (2).

For every $g \in S_L(r)$ we choose one geodesic $\mathfrak{q}_g$ joining it to 1. We thus obtain a set of geodesics of length r indexed by elements in $S_L(r)$. We denote this set of geodesics by $\mathcal{G}(r)$ and we identify it with the set $S_L(r)$.



Consider a geodesic $\mathfrak{q}_g$ in $\mathcal{G}(p)$, and an arbitrary geodesic triangle with $\mathfrak{q}_g$ as an edge and the other two edges $\mathfrak{q}_h \in \mathcal{G}(r_1)$ and $h\mathfrak{q}_k$, where $\mathfrak{q}_k \in \mathcal{G}(r_2)$. Such a triangle corresponds to a decomposition $g = hk$, where $h \in S_L(r_1)$ and $k \in S_L(r_2)$. Let us apply property (*) to this geodesic triangle. There exists γH_i, $i \in \{1, 2, \dots m\}$, such that $\overline{\mathcal{N}}_\sigma(\gamma H_i)$ intersects the three edges of the triangle. We denote as in the picture the respective entrance and exit points of the edges into $\overline{\mathcal{N}}_\sigma(\gamma H_i)$ by $A_1, A_2, B_1, B_2, C_1, C_2$.

We can write $g = g_1\eta g_2$, where $g_1 \in \gamma H_i$, g_1 is at distance at most σ from A_2, $\eta \in H_i$ and $g_1\eta$ is at distance at most σ from C_1. Property (*) implies that g_1 is at distance at most $\delta + \sigma$ from A_1 and that $g_1\eta$ is at distance at most $\delta + \sigma$ from C_2. It follows that there exists $\eta' \in H_i$ such that $g_1\eta'$ is at distance at most σ from B_2 and at distance at most $\delta + \sigma$ from B_1. If we denote by η'' the element $(\eta')^{-1}\eta$ then we have that $h = g_1\eta' g_3$ and that $k = g_3^{-1}\eta'' g_2$.

Notation: We denote by κ the constant $\sigma + \delta$.

The previous considerations justify the following notations and definitions.

Notations: We denote by Δ the set of all $(g_1, g_2, g_3, \eta, \eta', \eta'') \in G^3 \times \bigsqcup_{i=1}^m (H_i)^3$ such that:

(1) $g = g_1\eta g_2 \in S_L(p)$, $h = g_1\eta' g_3 \in S_L(r_1)$ and $k = g_3^{-1}\eta'' g_2 \in S_L(r_2)$;

(2) $\eta = \eta'\eta''$;

(3) If $(\eta, \eta', \eta'') \in (H_i)^3$ for some $i \in \{1, 2, \dots, m\}$, then the following hold:

- g_1 is at distance at most κ from the entrance point A_2 of $\mathfrak{q}_g$ into $\overline{\mathcal{N}}_\sigma(g_1 H_i)$ and from the entrance point A_1 of $\mathfrak{q}_h$ into $\overline{\mathcal{N}}_\sigma(g_1 H_i)$;
- $g_1\eta$ is at distance at most κ from the exit point C_1 of $\mathfrak{q}_g$ from $\overline{\mathcal{N}}_\sigma(g_1 H_i)$ and from the exit point C_2 of $\mathfrak{q}_k$ from $\overline{\mathcal{N}}_\sigma(g_1 H_i)$;
- $g_1\eta'$ is at distance at most κ from the exit point B_2 of $\mathfrak{q}_h$ from $\overline{\mathcal{N}}_\sigma(g_1 H_i)$ and from the entrance point B_1 of $\mathfrak{q}_k$ into $\overline{\mathcal{N}}_\sigma(g_1 H_i)$.

Definition 3.2. The set Δ is called the *set of central decompositions* of geodesic triangles with edges in $\mathcal{G}(p) \times \mathcal{G}(r_1) \times \mathcal{G}(r_2)$.

Definitions 3.3. Given a geodesic $\mathfrak{q}_g$ in $\mathcal{G}(r)$ with $r \in \{p, r_2\}$, we call every decomposition $g = g_1\eta g_2$ of g corresponding to a central decomposition in Δ a *central decomposition of* g. We call g_1 and g_2 the *left* and *right parts* of the decomposition; η is called the *middle part* of the decomposition of g.

For the fixed g (and $\mathfrak{q}_g$) we denote by $\mathcal{L}_g$ the set of left parts of central decompositions of g, by $\mathcal{R}_g$ the set of right parts of central decompositions of g and by $\mathcal{D}_g$ the set of triples (g_1, η, g_2) corresponding to central decompositions of g. We also denote by $\mathcal{LR}_g$ the set of pairs of left and right parts that can appear in a central decomposition of g.

We denote by $\mathcal{D}_p$ the set of all $\mathcal{D}_g$ with $g \in S_L(p)$. The sets $\mathcal{LR}_p$, $\mathcal{L}_p$, $\mathcal{R}_p$ are defined similarly.

Notations: Let C be a subset of $\prod_{i=1}^n X_i$. Let a_i be a point in the i-th projection of C, let I be a subset in $\{1, 2, \dots, n\} \setminus \{i\}$ and let $X^I = \prod_{i \in I} X_i$. We denote by $C^I(a_i)$ the set of elements $\bar{x}$ in X^I such that $(\bar{x}, a_i)$ occurs in the projection of C in $X^I \times X_i$. Whenever there is no risk of confusion, we drop the index I in $C^I(a_i)$.

For every fixed decomposition $d = (g_1, \eta, g_2) \in \mathcal{D}_g$ we also denote by C_d the set $\Delta(d)$, that is, the set of triples (η', η'', g_3) such that $(g_1, g_2, g_3, \eta, \eta', \eta'')$ is in Δ. We denote by $\mathcal{U}_d$ the set of projections on the last component of the triples in C_d, by $\mathcal{E}_d$ the set of projections (η', η'') on the first two components of the triples in C_d and by E'_d and E''_d the respective sets of η' and η''.

Lemma 3.4. *Every $g \in S_L(r)$, with $r \in \{p\,,\, r_2\}$, has at most $C_1 r_1 + C_2$ central decompositions, where C_1 and C_2 are universal constants.*

Proof. Suppose that g_1 is the left part of a central decomposition of g. Then g_1 is at distance at most κ from a point in a geodesic of length r_1. It follows that g_1 has length at most $r_1 + \kappa$. On the other hand g_1 is at distance at most κ from a vertex g'_1 on $\mathfrak{q}_g$, whence $L(g'_1) \leq r_1 + 2\kappa$. Thus, we have at most $r_1 + 2\kappa$ possibilities for g'_1. For each such g'_1 the number of left cosets γH_i at distance at most σ from it is bounded by an universal constant, so g'_1 can be the entrance point of $\mathfrak{q}_g$ in $\overline{\mathcal{N}}_\sigma(\gamma H_i)$ for at most a constant number of left cosets γH_i. The exit point g'_2 of $\mathfrak{q}_g$ from $\overline{\mathcal{N}}_\sigma(\gamma H_i)$ is uniquely defined each time the left coset is fixed. For each left coset, the number of points in it at distance at most σ from g'_1 is bounded by a universal constant, and likewise for g'_2. Thus, the number of possibilities for g_1 and $g_1\eta$, once g'_1 is fixed, is bounded by a constant. We deduce that on the whole there are at most $K(r_1 + 2\kappa)$ possible central decompositions of g, for some constant K. $\square$

We continue the proof of Theorem 3.1. We can write that

$$\|(x*y)_p\|^2 = \sum_{g\in S_L(p)} [(x*y)(g)]^2 = \sum_{g\in S_L(p)} \left[\sum_{(h,k)\in S_L(r_1)\times S_L(r_2), hk=g} x(h)y(k) \right]^2 \leq$$

$$\sum_{g\in S_L(p)} \left[\sum_{d\in\mathcal{D}_g} \sum_{(\eta',\eta'',g_3)\in C_d} x(g_1\eta' g_3) y(g_3^{-1}\eta'' g_2) \right]^2 . \tag{4}$$

We use the following easy consequence of the Cauchy-Schwartz inequality:

$$\left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2 \,. \tag{5}$$

This inequality, applied to the first sum in the brackets in the last term of (4), together with Lemma 3.4, gives

$$\|(x*y)_p\|^2 \leq (C_1 r_1 + C_2) \sum_{g\in S_L(p)} \sum_{d\in\mathcal{D}_g} \left[\sum_{(\eta',\eta'',g_3)\in C_d} x(g_1\eta' g_3) y(g_3^{-1}\eta'' g_2) \right]^2 . \tag{6}$$

We re-write the term in the brackets on the right hand side of inequality (6) and we apply the Cauchy-Schwartz inequality as follows:

$$\begin{aligned} \textstyle\sum_{(\eta',\eta'',g_3)\in C_d} x(g_1\eta' g_3) y(g_3^{-1}\eta'' g_2) = \sum_{e=(\eta',\eta'')\in\mathcal{E}_d} \sum_{g_3\in C_d(e)} x(g_1\eta' g_3) y(g_3^{-1}\eta'' g_2) \leq \\ \textstyle\sum_{e=(\eta',\eta'')\in\mathcal{E}_d} \left[\sum_{g_3\in C_d(e)} (x(g_1\eta' g_3))^2 \right]^{1/2} \left[\sum_{g_3\in C_d(e)} (y(g_3^{-1}\eta'' g_2))^2 \right]^{1/2} . \end{aligned} \tag{7}$$

We now define, for every $g \in S_L(p)$, $\bar{g} = (g_1, g_2) \in \mathcal{LR}_g$, $i \in \{1, 2, \ldots, m\}$ and for $\eta = g_1^{-1} g g_2^{-1} \in H_i$, the function $X^i_{\bar{g}} : H_i \to \mathbb{R}_+$ by

$$X^i_{\bar{g}}(\eta') = \left[\sum_{g_3 \in C_{(\bar{g},\eta)}(\eta'\,,\,(\eta')^{-1}\eta)} (x(g_1 \eta' g_3))^2 \right]^{1/2} \quad \text{for every } \eta' \in E'_{(\bar{g},\eta)}\,,$$

and $X^i_{\bar{g}}(\eta') = 0$ for all the other $\eta' \in H_i$.

Likewise we define the function $Y^i_{\bar{g}} : H_i \to \mathbb{R}_+$ by

$$Y^i_{\bar{g}}(\eta'') = \left[\sum_{g_3 \in C_{(\bar{g},\eta)}(\eta(\eta'')^{-1}\,,\,\eta'')} (y(g_3^{-1} \eta'' g_2))^2 \right]^{1/2} \quad \text{for every } \eta'' \in E''_{(\bar{g},\eta)}\,,$$

and $Y^i_{\bar{g}}(\eta'') = 0$ for all the other $\eta'' \in H_i$.

Then the sum in (7), if the middle part η of d is in H_i, can be written as

$$\sum_{(\eta',\eta'') \in \mathcal{E}_d \cap H_i \times H_i} X^i_{\bar{g}}(\eta') Y^i_{\bar{g}}(\eta'')\,.$$

Therefore inequality (6) gives

$$\|(x * y)_p\|^2 \leq$$

$$(C_1 r_1 + C_2) \sum_{g \in S_L(p)} \sum_{\bar{g} \in \mathcal{LR}_g} \sum_{i=1}^{m} \sum_{\eta \in \mathcal{D}_g(\bar{g}) \cap H_i} \left[\sum_{(\eta',\eta'') \in \mathcal{E}_{(\bar{g},\eta)} \cap H_i \times H_i} X^i_{\bar{g}}(\eta') Y^i_{\bar{g}}(\eta'') \right]^2 . \tag{8}$$

We denote by S the sum in the second term of the inequality (8), without the factor $C_1 r_1 + C_2$. We have

$S \leq \sum_{g \in S_L(p)} \sum_{\bar{g} \in \mathcal{LR}_g} \sum_{i=1}^{m} \|X^i_{\bar{g}} * Y^i_{\bar{g}}\|^2 \leq \sum_{g \in S_L(p)} \sum_{\bar{g} \in \mathcal{LR}_g} \sum_{i=1}^{m} P_i(r_1 + 2\kappa)^2 \|X^i_{\bar{g}}\|^2 \|Y^i_{\bar{g}}\|^2 \leq$

$P(r_1) \sum_{g \in S_L(p)} \sum_{\bar{g} \in \mathcal{LR}_g} \sum_{i=1}^{m} \|X^i_{\bar{g}}\|^2 \|Y^i_{\bar{g}}\|^2$.

The latter sum is smaller than

$$\sum_{\bar{g} \in \mathcal{LR}_p} \left[\sum_{\eta \in \mathcal{D}_p(\bar{g}), \eta' \in E'_{(\bar{g},\eta)}} \sum_{g_3 \in C_{(\bar{g},\eta)}(\eta',(\eta')^{-1}\eta)} (x(g_1 \eta' g_3))^2 \right] \cdot \left[\sum_{\eta \in \mathcal{D}_p(\bar{g}), \eta'' \in E''_{(\bar{g},\eta)}} \sum_{g_3 \in C_{(\bar{g},\eta)}(\eta(\eta'')^{-1},\eta'')} (y(g_3^{-1} \eta'' g_2))^2 \right] \leq$$

$$\left[\sum_{g_1 \in \mathcal{L}_p} \sum_{(\eta', g_3) \in \Delta(g_1)} (x(g_1 \eta' g_3))^2 \right] \cdot \left[\sum_{g_2 \in \mathcal{R}_p} \sum_{(g_3, \eta'') \in \Delta(g_2)} (y(g_3^{-1} \eta'' g_2))^2 \right] .$$

Every $x(g)^2$ with $g \in S_L(r_1)$ appears at most $K_1 r_1$ times in the first sum above, where K_1 is an universal constant, hence the first sum is at most $K_1 r_1 \|x\|^2$. Lemma 3.4 applied to every $g \in S_L(r_2)$ implies that every $y(g)^2$ with $g \in S_L(r_2)$ appears at most $C_1 r_1 + C_2$ times in the second sum above, hence the second sum is at most $(C_1 r_1 + C_2)\|y\|^2$.

We deduce that

$$S \leq Q_1(r_1) P(r_1) \|x\|^2 \|y\|^2\,,$$

where Q_1 is an universal polynomial of degree 2. We may conclude that

$$\|(x * y)_p\|^2 \leq Q(r_1) P(r_1) \|x\|^2 \|y\|^2\,,$$

where Q is an universal polynomial of degree 3. □

Remark 3.5. Theorem 3.1 holds if we replace property (*) in the definition of (*)-relative hyperbolicity by the following weaker property.

We say that a countable group G endowed with a length-function L is (**)-relatively hyperbolic with respect to the subgroups $H_1, ..., H_m$ if there exist two polynomials $Q_1(r)$ and $Q_2(r)$ such that the following condition holds.

(**) Let $\mathcal{H} = \bigcup_{i=1}^m (H_i \times H_i)$. There exists a map $T\colon G \times G \to G \times \mathcal{H}$ such that
(i) if $T(g, h) = (a, g', h')$ then $T(h, g) = (a, h', g')$ and

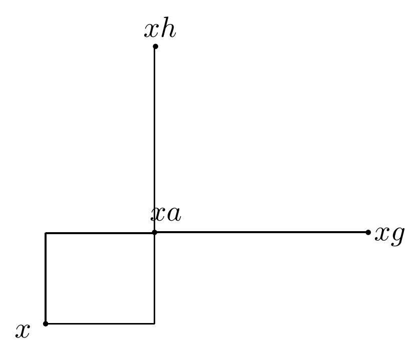
$$T(h^{-1}, h^{-1}g) = (h^{-1}ah', (h')^{-1}, (h')^{-1}g');$$

(ii) $L(h') \leq Q_1(L(h))$;

(iii) for each $g \in G$ the number of pairs (a, g') such that $(a, g', \cdot\,) = T(g, h)$ for some h with $L(h) = r$ does not exceed $Q_2(r)$.

One can interpret a pair (g, h) as two sides of the triangle with vertices x, xg, xh (for any basepoint $x \in G$). The image $T(g, h)$ gives three vertices of a triangle xa, xag', xah' from a coset of H_i. Condition (i) means that the set $\{a, ag', ah'\}$ is stable under permutations of the vertices x, xg, xh. Condition (ii) means that the sides of the resulting triangle are not too large. The pair (a, g') can be interpreted as a central decomposition of g corresponding to the triangle $(1, g, h)$ (more precisely the left and the middle part of the decomposition, the right part is uniquely determined by the other two). Condition (iii) means that the number of central decompositions of g corresponding to triangles (x, xg, xh) with $L(h) = r$ is bounded by a polynomial in r. Condition (iii) replaces Lemma 3.4, the proof of Theorem 3.1 carries almost verbatim.

The class of (**)-relatively hyperbolic groups is certainly wider than the class of relatively hyperbolic groups. For example $\mathbb{Z}^2$ is (**)-relatively hyperbolic with respect to the trivial subgroup. To construct a map T, consider the two Γ-shaped geodesics in $\mathbb{Z}^2$ connecting any two points A, B. For every pair $\{g, h\}$ choose one of the two possible geodesics for each pair $(1, g)$, $(1, h)$, (g, h) so that the three geodesics intersect in a point a. The map T takes (g, h) to $(a, 1, 1)$. One should choose the point a so that condition (i) is satisfied: if we choose a certain a for the pair (g, h) then the corresponding a's for (h, g) and $(h^{-1}, h^{-1}g)$ are determined uniquely. Then the conditions (ii) and (iii) are also satisfied, for $Q_1(r) = 0$, $Q_2(r) = 2r$.



Similarly, for every k, $\mathbb{Z}^k$ is (**)-relatively hyperbolic with respect to the trivial subgroup.

More generally, every group with polynomial growth function $f(n)$ is (**)-relatively hyperbolic with respect to the trivial subgroup: for every pair g, h pick a shortest side of the triangle with vertices $1, g, h$. Let a be g if g is a vertex of the chosen shortest side, or 1 otherwise (again

choose a depending on (g, h) so that (i) is satisfied). The map T takes (g, h) to $(a, 1, 1)$. It is an easy exercise to check that conditions (ii) and (iii) are satisfied with $Q_1(r) = 0$, $Q_2(r) = 2f(r)+2$.

A group G that is (*)-relatively hyperbolic (in particular relatively hyperbolic) with respect to the subgroups $H_1, ..., H_m$ is (**)-relatively hyperbolic with respect to these subgroups, by Lemma 3.4 (both Q_1 and Q_2 are linear polynomials).

It would be interesting to see how large the class of (**)-relatively hyperbolic groups is. In particular, it would be interesting to know if every mapping class group is (**)-relatively hyperbolic with respect to direct products of mapping class groups of smaller genus (in that case we would be able to proceed by induction and prove that all mapping class groups have RD).

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